GLOBAL CONSERVATIVE SOLUTIONS OF THE CAMASSA–HOLM EQUATION FOR INITIAL DATA WITH NONVANISHING ASYMPTOTICS

KATRIN GRUNERT, HELGE HOLDEN, AND XAVIER RAYNAUD

Abstract. We study global conservative solutions of the Cauchy problem for the Camassa–Holm equation $u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$ with nonvanishing and distinct spatial asymptotics.

1. Introduction

The Cauchy problem for the Camassa–Holm (CH) equation [5, 6],

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.1}$$

where $\kappa \in \mathbb{R}$ is a constant, has attracted much attention due to the fact that it serves as a model for shallow water waves [9] and its rich mathematical structure. For example, it has a bi-Hamiltonian structure, infinitely many conserved quantities and blow-up phenomena have been studied. As these properties play no role in the present approach, we refer to [12] for references to papers that discuss these properties.

In particular, global conservative solutions have been constructed in the periodic case [15] and on the real line in the case of initial data with the same vanishing asymptotics at minus and plus infinity [2, 12] (for $\kappa = 0$) and [13] (for $\kappa \neq 0$). Furthermore, a Lipschitz metric has been derived for the Camassa–Holm equation [10, 11].

Here we focus on the construction of a semigroup of global conservative solutions on the real line for initial data with (in general) different asymptotics at minus and plus infinity. The approach used here is similar to the one used in [12] for the Camassa–Holm equation in the case of vanishing asymptotics (when $u \in H^1(\mathbb{R})$). It also resembles a recent study of the Hunter–Saxton equation in [4], and indeed we here combine the two approaches. In [1], the authors proves in the dissipative case the existence of H^1 perturbations around a given solution with nonvanishing asymptotics. In this article, by solving the Cauchy problem, we prove that such solutions exist in the conservative case.

More precisely, we consider the initial problem for (1.1) with initial data $u_0 \in H_\infty(\mathbb{R})$, which means that u_0 can be written as

$$u_0(x) = \bar{u}_0(x) + c_-\chi(-x) + c_+\chi(x), \tag{1.2}$$

Date: June 22, 2011.
2010 *Mathematics Subject Classification.* Primary: 35Q53, 35B35; Secondary: 35Q20.
Key words and phrases. Camassa–Holm equation, conservative solutions, nonvanishing asymptotics.
Research supported in part by the Research Council of Norway under projects Wavemaker, NoPiMa, and by the Austrian Science Fund (FWF) under Grant No. Y330.

for some constants $c_\pm \in \mathbb{R}$, where $\bar{u}_0 \in H^1(\mathbb{R})$ and χ denotes a smooth partition function, which satisfies $\chi(x) = 0$ for $x \leq 0$, $\chi(x) = 1$ for $x \geq 1$ and $\chi'(x) \geq 0$ for all $x \in \mathbb{R}$.

In [16, 17], traveling waves solutions of the CH equation have been characterized and depicted. The solutions are obtained by glueing together simpler solutions. In particular, Lenells constructs solutions with distinct asymptotics at plus and minus infinity, see Figure 1 for an example.

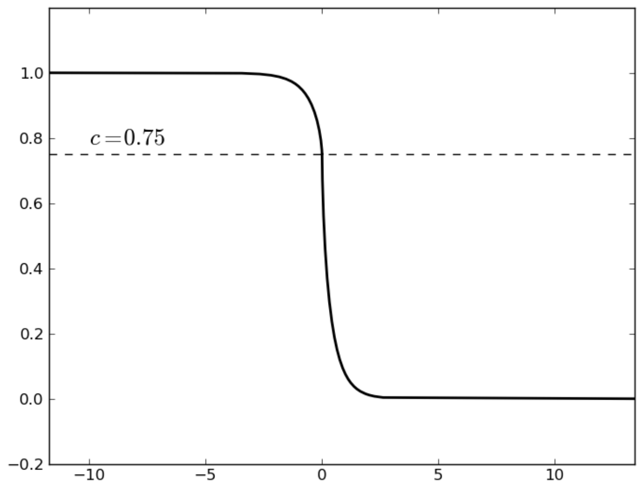

FIGURE 1. Traveling wave $u(t,x) = \phi(x - ct)$ obtained by gluing together two *cuspons* at the point $(0, c)$. For the picture here, we use $c = \frac{3}{4}$ and $k = -\frac{7}{8}$ so that the gluing indeed results in a weak solution (see [16]). Note that $\phi_x \in L^2(\mathbb{R})$ but ϕ is not Lipschitz as $\lim_{x\to 0} \phi_x(x) = -\infty$.

Without loss of generality, we can consider the case $\kappa = 0$ (see Section 2). Then (1.1) can be rewritten as the following system

$$u_t + uu_x + P_x = 0, \tag{1.3a}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{1.3b}$$

The last equation yields, after using the Green function,

$$P(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(u^2 + \frac{1}{2}u_x^2)(z)\, dz$$

and we can see that P is well-defined when $u \in H_\infty(\mathbb{R})$. We show that the asymptotic values c_- and c_+ for $u(t,x)$ at infinity are in fact constant in time (see Section 2). This may seem a little surprising since the Camassa–Holm equation with vanishing asymptotics has infinite speed of propagation [7], and therefore one might think that the solution would approach a common asymptotic, e.g., the mean value, at plus and minus infinity.

The solutions of the Camassa–Holm equation may experience wave breaking (see, e.g., [8] and references therein). The continuation of the solution past wave breaking is highly nontrivial, and allows for several distinct continuations. Two prominent classes of solutions are denoted as conservative and dissipative solutions, see, e.g., [2, 3, 12, 14].

The aim of this article is to construct a semigroup of conservative solutions on the line for nonvanishing asymptotics. In the case of vanishing asymptotics, conservative solutions refer to the solutions for which the $H^1(\mathbb{R})$ norm is preserved, for almost every time. Here, it does not make sense to consider the $H^1(\mathbb{R})$ norm of the solution. By conservative solutions, we mean weak solutions to the equation which in addition satisfy the conservation law

$$(u^2+u_x^2)_t+(u(u^2+u_x^2))_x=(u^3-2Pu)_x$$

in the sense of distributions (see Definition (5.1) for the precise definition).

The continuity of the semigroup is obtained with respect of a new metric that we introduce. This metric depends on the choice of the partition function χ. However, in Section 6, it is shown that different choices of partition functions χ lead to the same topology.

2. Eulerian coordinates

We consider the Cauchy problem for the Camassa–Holm equation with arbitrary $\kappa \in \mathbb{R}$ given by

$$u_t-u_{txx}+\kappa u_x+3uu_x-2u_xu_{xx}-uu_{xxx}=0, \quad u|_{t=0}=u_0. \tag{2.1}$$

We are interested in global solutions for initial data with nonvanishing limits at infinity, that is,

$$\lim_{x\to-\infty}u_0(x)=u_{-\infty} \quad \text{and} \quad \lim_{x\to\infty}u_0(x)=u_{\infty}. \tag{2.2}$$

To be more specific, we assume that $u_0(x)$ can be rewritten as

$$u_0(x)=\bar{u}_0(x)+u_{-\infty}\chi(-x)+u_{\infty}\chi(x), \tag{2.3}$$

with $\bar{u}_0\in H^1(\mathbb{R})$ and χ a smooth partition function, which has support in $[0,\infty)$, satisfies $\chi(x)=1$ for $x\geq 1$ and $\chi'(x)\geq 0$ for $x\in\mathbb{R}$. Assuming that $u(t,x)$ satisfies (1.1), then the function $v(t,x)=u(t,x-\kappa t/2)+\kappa/2$ satisfies (1.1) with $\kappa=0$ and hence we can without loss of generality assume that $\kappa=0$, because the framework presented here allows for nonvanishing asymptotics.

Let us introduce the mapping I_χ from $H^1(\mathbb{R})\times\mathbb{R}^2$ into $H^1_{\rm loc}(\mathbb{R})$ given by

$$I_\chi(\bar{u},c_-,c_+)=\bar{u}(x)+c_-\chi(-x)+c_+\chi(x)$$

for any $(\bar{u},c_-,c_+)\in H^1(\mathbb{R})\times\mathbb{R}^2$. We denote by $H_\infty(\mathbb{R})$ the image of $H^1(\mathbb{R})\times\mathbb{R}^2$ by I_χ, that is, $H_\infty(\mathbb{R})=I_\chi(H^1(\mathbb{R})\times\mathbb{R}^2)$. Then, u_0 as given by (2.3) belongs by definition to $H_\infty(\mathbb{R})$. Since I_χ is linear, $H_\infty(\mathbb{R})$ is a vector space. The mapping I_χ is injective. We equip $H_\infty(\mathbb{R})$ with the norm

$$\|u\|_{H_\infty(\mathbb{R})}=\|\bar{u}\|_{H^1(\mathbb{R})}+|c_-|+|c_+| \tag{2.4}$$

where $u=I_\chi(\bar{u},c_-,c_+)$. Then, $H_\infty(\mathbb{R})$ is a Banach space. Given another partition function $\tilde{\chi}$, we define the mapping $(\tilde{\bar{u}},\tilde{c}_-,\tilde{c}_+)=\Psi(\bar{u},c_-,c_+)$ from $H^1(\mathbb{R})\times\mathbb{R}^2$ to $H^1(\mathbb{R})\times\mathbb{R}^2$ as $\tilde{c}_-=c_-$, $\tilde{c}_+=c_+$ and

$$\tilde{\bar{u}}(x)=\bar{u}(x)+c_-(\chi(-x)-\tilde{\chi}(-x))+c_+(\chi(x)-\tilde{\chi}(x)). \tag{2.5}$$

The linear mapping Ψ is a continuous bijection. Since

$$I_\chi = I_{\tilde{\chi}} \circ \Psi,$$

we can see that the definition of the Banach space $H_\infty(\mathbb{R})$ does not depend on the choice of the partition function χ. The norm defined by (2.4) for different partition functions χ are all equivalent.

If $u(t,x)$ is a solution of the Camassa–Holm equation, then, for any constants α and β, we easily find that

$$v(t,x) = \alpha u(\alpha t, x - \beta t) + \beta \tag{2.6}$$

is also a solution with κ replaced by $\alpha\kappa - 2\beta$. The case $u_\infty = u_{-\infty}$, can be reduced to the standard case of vanishing asymptotics at infinity by choosing $\alpha = 1$ and $\beta = -u_\infty$. Furthermore, in the case when $u_\infty \neq u_{-\infty}$, it is no loss of generality to only consider initial conditions which satisfy the following conditions

$$\lim_{x\to-\infty} u_0(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} u_0(x) = c, \tag{2.7}$$

where c denotes some constant. Especially, if u_0 is an initial condition which satisfies (2.2), then take $\alpha = 1$ and $\beta = -u_{-\infty}$ in (2.6) and we obtain an initial condition v_0 which satisfies condition (2.7). Accordingly, we introduce the subspace $H_{0,\infty}(\mathbb{R})$ of $H_\infty(\mathbb{R})$ as

$$H_{0,\infty}(\mathbb{R}) = I_\chi(H^1(\mathbb{R}) \times \{0\} \times \mathbb{R}).$$

As we shall see next this subspace is preserved by the Camassa–Holm equation. This is the main motivation for considering this space beside of the fact that the arguments simplify.

In what follows we will restrict ourselves to the case $\kappa = 0$, as the case $\kappa \neq 0$ can be treated using the same ideas and techniques. For the case $\kappa = 0$ the governing equations read[1]

$$u_t + uu_x + P_x = 0, \tag{2.8a}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{2.8b}$$

Let us assume that $\lim_{x\to\infty} u(t,x) = c(t)$ exists. From (2.8b) we obtain the following representation for P, under the assumption that $u \in H_{0,\infty}(\mathbb{R})$,

$$P(x) = c^2\chi^2(x) + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(2c\chi\bar{u} + \bar{u}^2 + \frac{1}{2}u_x^2 + 2c^2\chi'^2 + 2c^2\chi\chi'')(y)dy. \tag{2.9}$$

It follows that

$$P_x(x) = 2c^2\chi'(x)\chi(x) - \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x-y)e^{-|x-y|}(2c\chi\bar{u} + \bar{u}^2 + \frac{1}{2}u_x^2 + 2c^2\chi'^2 + 2c^2\chi\chi'')(y)dy$$

and $\lim_{x\to\infty} P_x = 0$. Thus, we formally obtain that $c'(t) = \lim_{x\to\infty} u_t(t,x) = 0$ and the limit at infinity of u is constant in time. Indeed the solutions we are going to construct satisfy this property.

[1] For κ nonzero (2.8b) is simply replaced by $P - P_{xx} = u^2 + \kappa u + \frac{1}{2}u_x^2$.

3. Lagrangian variables

The aim of this section is to rewrite the Camassa–Holm equation as a system of ordinary differential equations, which describes solutions in Lagrangian coordinates. Let V be the Banach space

$$V = \{f \in C_b(\mathbb{R}) \mid f_\xi \in L^2(\mathbb{R})\},$$

where $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, equipped with the norm

$$\|f\|_V = \|f\|_{L^\infty} + \|f_\xi\|_{L^2}\,, \tag{3.1}$$

for any $f \in V$. Then we define $y_t(t,\xi) = u(t, y(t,\xi))$, which can be rewritten as $y(\xi) = \zeta(\xi) + \xi$, where ζ belongs to V. Furthermore we set $U(t,\xi) = u(t, y(t,\xi))$, which can be decomposed as

$$U = \bar{U} + c\chi \circ y$$

where $\bar{U} \in H^1(\mathbb{R})$ and $c \in \mathbb{R}$. We define $h \in L^2(\mathbb{R})$ formally as

$$h(t,\xi) = u_x^2(t, y(t,\xi))y_\xi(t,\xi)$$

so that $u_x^2(t,x)\,dx = y_\#(h(t,\xi)\,d\xi)$. Here $y_\#$ denotes the push-forward.[2] We have

$$y_\xi h = U_\xi^2. \tag{3.2}$$

Next we derive the equivalent system for the independent variables ζ, U, and h. Therefore set $P(t,\xi) = P(t, y(t,\xi))$, where $P(t,x)$ is the solution of (2.8b) and define $Q(t,\xi) = P_x(t, y(t,\xi))$, then we have using (2.8a)

$$\begin{aligned} y_t &= U, \\ U_t &= -Q. \end{aligned}$$

Let us compute h_t. Assuming that the solution is smooth, (2.8) yields

$$(u_x^2)_t + (uu_x^2)_x = 2(u^2 u_x - Pu_x). \tag{3.3}$$

By the definition of y as the characteristic function, it follows that

$$\frac{\partial}{\partial t}(u_x^2 \circ y y_\xi) = ((u_x^2)_t + (uu_x^2)_x) \circ y y_\xi. \tag{3.4}$$

Hence, from (3.3), we get

$$h_t = 2((u^2 - P) \circ y)(u_x \circ y)y_\xi = 2(U^2 - P)U_\xi. \tag{3.5}$$

Thus, we consider the system

$$y_t = U, \tag{3.6a}$$

$$U_t = -Q, \tag{3.6b}$$

$$h_t = 2(U^2 - P)U_\xi. \tag{3.6c}$$

After studying the functions P and Q we will prove the local existence of solutions to (3.6) in

$$E := V \times H_{0,\infty}(\mathbb{R}) \times L^2(\mathbb{R})$$

by using a standard contraction argument. The norm of E is given in terms of a partition function χ. Then E is in isometry with

$$\bar{E} = V \times H^1(\mathbb{R}) \times \mathbb{R} \times L^2(\mathbb{R}).$$

[2]The push-forward of a measure ν by a measurable function f is the measure $f_\#\nu$ defined as $f_\#\nu(B) = \nu(f^{-1}(B))$ for any Borel set B.

We have

$$\|(\zeta, U, h)\|_E = \left\|(\zeta, I_\chi^{-1}(U), h)\right\|_{\bar{E}}.$$

However, as noted earlier, all partition functions give rise to equivalent norms. For convenience, we will often abuse the notations and denote by the same X the two elements $(\zeta, \bar{U}, c, h)$ and (y, U, h) where, by definition, $U = \bar{U} + c\chi \circ y$ and $y(\xi) = \zeta(\xi) + \xi$.

Lemma 3.1. *For any $X = (\zeta, U, h)$ in E, we define the maps $\mathcal{Q}$ and $\mathcal{P}$ as $\mathcal{Q}(X) = Q$ and $\mathcal{P}(X) = P$, where P and Q are given by*

$$P(\xi) = \frac{1}{4}\int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|}((2\bar{U}^2 + 4c\bar{U}\chi \circ y)y_\xi + h)(\eta)\,d\eta + c^2 g \circ y(\xi), \tag{3.7}$$

and

$$Q(\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(\xi - \eta) e^{-|y(\xi)-y(\eta)|}((2\bar{U}^2 + 4c\bar{U}\chi \circ y)y_\xi + h)(\eta)\,d\eta + c^2 g' \circ y(\xi), \tag{3.8}$$

where

$$g(x) = \chi^2(x) + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(2\chi'^2 + 2\chi\chi'')(z)\,dz. \tag{3.9}$$

Then, $X \mapsto \mathcal{P} - U^2$ and $X \mapsto \mathcal{Q}$ are locally Lipschitz maps, i.e., Lipschitz on bounded sets, from E to $H^1(\mathbb{R})$. Moreover we have

$$Q_\xi = -\frac{1}{2}h - (U^2 - P)y_\xi, \tag{3.10}$$

$$P_\xi = Q(1 + \zeta_\xi). \tag{3.11}$$

Proof. The expression (3.7) is obtained from (2.8b) after a change of variables to Lagrangian variables. From (2.8b), we get

$$P - P_{xx} = c^2\chi^2 + 2c\chi\bar{u} + \bar{u}^2 + \frac{1}{2}u_x^2,$$

which yields, after applying the Helmholtz operator,

$$P(x) = c^2 g(x) + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(2c\chi\bar{u} + \bar{u}^2 + \frac{1}{2}u_x^2)(y)dy, \tag{3.12}$$

where we define g as the solution of $g - g'' = \chi^2$. Since $(g - \chi^2) - (g - \chi^2)_{xx} = 2\chi''\chi + 2\chi'^2$, after applying the Helmholtz operator, we get

$$g - \chi^2 = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(2\chi'^2 + 2\chi\chi'')(z)\,dz \tag{3.13}$$

and we recover the definition (3.9). The integral term on the right-hand side in (3.13) belongs to $H^1(\mathbb{R})$, and thus it follows that $\lim_{x\to-\infty} g(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$. Moreover, since we can also write $g(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|z|}\chi^2(x-z)\,dz$, we have

$$g'(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}2\chi'(z)\chi(z)\,dz \tag{3.14}$$

so that $g' \geq 0$, as $\chi' \geq 0$. Thus,

$$\|g\|_{L^\infty} = \lim_{x\to\infty} g(x) = 1.$$

Defining now $P(t,\xi) = P(t, y(t,\xi))$, then (3.12) yields (3.7), after changing variable and using (3.2). Analogously one explains the definition (3.8).

Next we prove that $\mathcal{Q}$ is locally Lipschitz from E to $H^1(\mathbb{R})$. We rewrite $\mathcal{Q}$ as

$$\begin{aligned}\mathcal{Q}(X)(\xi) &= -\frac{e^{-\zeta(\xi)}}{4}\int_{-\infty}^{\xi} e^{-|\xi-\eta|}e^{\zeta(\eta)}[(2\bar{U}^2+4c\bar{U}\chi\circ y)y_\xi+h]d\eta\\ &\quad+\frac{e^{\zeta(\xi)}}{4}\int_{\xi}^{\infty} e^{-|\xi-\eta|}e^{-\zeta(\eta)}[(2\bar{U}^2+4c\bar{U}\chi\circ y)y_\xi+h]d\eta\\ &\quad+c^2g'\circ y\\ &=\mathcal{Q}_1+\mathcal{Q}_2+c^2g'\circ y.\end{aligned}$$

Let $f(\xi)=\chi_{\xi>0}(\xi)e^{-\xi}$ and A be the map defined by $A\colon v\mapsto f\star v$. Then $\mathcal{Q}_1$ can be rewritten as

$$\mathcal{Q}_1(X)(\xi) = -\frac{e^{-\zeta(\xi)}}{4}A\circ R(\zeta,U,h)(\xi), \tag{3.15}$$

where R is the operator from E to $L^2(\mathbb{R})$ given by

$$R(\zeta,U,h)=e^{\zeta}\Big((2\bar{U}^2+4c\bar{U}\chi\circ y)y_\xi+h\Big). \tag{3.16}$$

The Fourier transform of f can be easily computed, and we obtain

$$\hat{f}(\eta)=\int_{\mathbb{R}} f(\xi)e^{-2\pi\mathrm{i}\eta\xi}d\xi=\frac{1}{1+2\pi\mathrm{i}\eta}. \tag{3.17}$$

The $H^1(\mathbb{R})$-norm can be expressed in terms of the Fourier transform as follows

$$\begin{aligned}\|f\star v\|_{H^1} &= \Big\|(1+\eta^2)^{\frac12}\widehat{f\star v}\Big\|_{L^2}\\ &=\Big\|(1+\eta^2)^{\frac12}\hat{f}\hat{v}\Big\|_{L^2}\\ &\le C\,\|\hat{v}\|_{L^2}\\ &=C\,\|v\|_{L^2},\end{aligned}$$

for some constant C. Hence $A: L^2(\mathbb{R})\to H^1(\mathbb{R})$ is continuous. Let us prove that $\mathcal{Q}_1$ is locally Lipschitz from E to $H^1(\mathbb{R})$. It is not hard to prove that R is locally Lipschitz from E to $L^2(\mathbb{R})$, by applying

$$|\chi\circ y_1(\xi)-\chi\circ y_2(\xi)|=|\int_{y_1(\xi)}^{y_2(\xi)}\chi'(x)dx|\le\|\chi'\|_{L^\infty}|y_2(\xi)-y_1(\xi)|,$$

as the following estimate shows

$$\begin{aligned}&\big\|e^{\zeta_1}[(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}+h_1]-e^{\zeta_2}[(2\bar{U}_2^2+4c_2\bar{U}_2\chi\circ y_2)y_{2,\xi}+h_2]\big\|_{L^2}\\ &\le\big\|(e^{\zeta_1}-e^{\zeta_2})[(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}+h_1]\big\|_{L^2}\\ &\quad+\big\|e^{\zeta_2}[(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}-(2\bar{U}_2^2+4c_2\bar{U}_2\chi\circ y_2)y_{2,\xi}+h_1-h_2]\big\|_{L^2}\\ &\le\big\|e^{\zeta_1}-e^{\zeta_2}\big\|_{L^\infty}\big\|(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}+h_1\big\|_{L^2}\\ &\quad+\big\|e^{\zeta_2}\big\|_{L^\infty}\big\|(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}-(2\bar{U}_2^2+4c_2\bar{U}_2\chi\circ y_2)y_{2,\xi}+h_1-h_2\big\|_{L^2}\\ &\le e^{\|\zeta_1\|_{L^\infty}+\|\zeta_2\|_{L^\infty}}\|\zeta_1-\zeta_2\|_{L^\infty}\big\|(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}+h_1\big\|_{L^2}\\ &\quad+e^{\|\zeta_2\|_{L^\infty}}(\|h_1-h_2\|_{L^2}+4\big\|c_1\bar{U}_1\chi\circ y_1-c_2\bar{U}_2\chi\circ y_2\big\|_{L^2}+2\big\|\bar{U}_1^2\zeta_{1,\xi}-\bar{U}_2^2\zeta_{2,\xi}\big\|_{L^2}\\ &\qquad+4\big\|c_1\bar{U}_1\chi\circ y_1\zeta_{1,\xi}-c_2\bar{U}_2\chi\circ y_2\zeta_{2,\xi}\big\|_{L^2}+2\big\|\bar{U}_1^2-\bar{U}_2^2\big\|_{L^2})\\ &\le\|y_1-y_2\|_{L^\infty}e^{\|\zeta_1\|_{L^\infty}+\|\zeta_2\|_{L^\infty}}\big\|(2\bar{U}_1^2+4c_1\bar{U}_1\chi\circ y_1)y_{1,\xi}+h_1\big\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+ e^{\|\zeta_2\|_{L^\infty}} \Big(\|h_1 - h_2\|_{L^2} + 2(\|\bar U_1\|_{L^\infty} + \|\bar U_2\|_{L^\infty}) \|\bar U_1 - \bar U_2\|_{L^2} + 4|c_1 - c_2| \|\bar U_1\|_{L^2} \\
&\qquad + |c_2| \|\bar U_1 - \bar U_2\|_{L^2} + |c_2| \|\bar U_2\|_{L^2} \|\chi \circ y_1 - \chi \circ y_2\|_{L^\infty} \\
&\qquad + 2 \|\zeta_{1,\xi}\|_{L^2} \|\bar U_1 - \bar U_2\|_{L^\infty} \|\bar U_1 + \bar U_2\|_{L^\infty} \\
&\qquad + 2 \|\bar U_2\|_{L^\infty}^2 \|\zeta_{1,\xi} - \zeta_{2,\xi}\|_{L^2} + \|\zeta_{1,\xi} - \zeta_{2,\xi}\|_{L^2} \|4c_1 \bar U_1 \chi \circ y_1\|_{L^\infty} \\
&\qquad + 4 \|\zeta_{2,\xi}\|_{L^2} \|c_1 \bar U_1 \chi \circ y_1 - c_2 \bar U_2 \chi \circ y_2\|_{L^\infty} \Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\|c_1 \bar U_1 \chi \circ y_1 - c_2 \bar U_2 \chi \circ y_2\|_{L^\infty} \leq{}& |c_1 - c_2| \|\bar U_1\|_{L^\infty} + |c_2| \|\bar U_1 - \bar U_2\|_{L^\infty} \\
&+ C|c_2| \|\bar U_2\|_{L^\infty} \|y_1 - y_2\|_{L^\infty}.
\end{aligned}
$$

Since A is linear and continuous from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$, the composition $A \circ R$ is locally Lipschitz from E to $H^1(\mathbb{R})$. Then, we use the following lemma, which is stated without proof.

Lemma 3.2. *Let $\mathcal{R}_1 \colon E \to V$ and $\mathcal{R}_2 : E \to H^1(\mathbb{R})$, or $\mathcal{R}_2 : E \to V$ be two locally Lipschitz maps. Then the product $X \to \mathcal{R}_1(X)\mathcal{R}_2(X)$ is also locally Lipschitz from E to $H^1(\mathbb{R})$, or from E to V.*

Since the mapping $X \mapsto e^{-\zeta}$ is locally Lipschitz from E to V, the function $\mathcal{Q}_1$ is the product of two locally Lipschitz maps, one from E to $H^1(\mathbb{R})$ and the other one from E to V, it is locally Lipschitz from E to $H^1(\mathbb{R})$. Similarly one proves that $\mathcal{Q}_2$ is locally Lipschitz. Thus it is left to show that $X \mapsto g' \circ y$ is locally Lipschitz from E to $H^1(\mathbb{R})$. By (3.14) we have

$$
\begin{aligned}
g'(y(\xi)) &= \int_{-\infty}^{\xi} e^{-(y(\xi)-y(z))} \chi'(y(z))\chi(y(z)) y_\xi(z) dz \\
&\quad + \int_{\xi}^{\infty} e^{-(y(z)-y(\xi))} \chi'(y(z))\chi(y(z)) y_\xi(z) dz \\
&= I_1(\xi) + I_2(\xi).
\end{aligned}
$$

Introduce $v(z) = e^{\zeta(z)} \chi'(y(z))\chi(y(z)) y_\xi(z)$, then we can write $I_1(\xi)$ as

$$
I_1(\xi) = e^{-\zeta(\xi)} A(v) \tag{3.18}
$$

and we only have to check that the mapping $X \mapsto v$ is locally Lipschitz from E to $L^2(\mathbb{R})$. This follows from the smoothness of χ and the fact that

$$
\begin{aligned}
\|\chi'(y(\xi))\|_{L^2} &\leq \|\chi'\|_{L^\infty} (\operatorname{meas}\{\xi \in \mathbb{R} \mid y(\xi) \in [0,1]\})^{1/2} \\
&\leq \|\chi'\|_{L^\infty} (\operatorname{meas}\{[-\|\zeta\|_{L^\infty}, 1 + \|\zeta\|_{L^\infty}]\})^{1/2} \\
&\leq C.
\end{aligned}
$$

Therefore $\mathcal{Q}$ is locally Lipschitz from E to $H^1(\mathbb{R})$. To prove that $\mathcal{P} - U^2$ is locally Lipschitz from E to $H^1(\mathbb{R})$ one can use the same techniques after discovering that one can write, using (3.9),

$$
\begin{aligned}
P(\xi) - U(\xi)^2 &= \frac{1}{4} \int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|} \Big((2\bar U^2 + 4c\bar U \chi(y)) y_\xi + h \Big)(\eta) d\eta \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|} \Big(2c^2 (\chi'(y))^2 + 2c^2 \chi(y)\chi''(y) \Big) y_\xi(\eta) d\eta
\end{aligned}
$$

$$- \bar{U}(\xi)^2 - 2\bar{U}(\xi)c\chi(y(\xi)).$$

Hence $\mathcal{Q}$ and $\mathcal{P} - U^2$ are locally Lipschitz continuous from E to $H^1(\mathbb{R})$. ☐

By the above lemma we have that $Q \in H^1(\mathbb{R})$ and therefore $\lim_{\xi\to\pm\infty} Q(t,\xi) = 0$. Hence (3.6) implies that if a solution in E exists the asymptotic behavior of $U(t,\xi)$ must be preserved for all times. Thus using that $y - \mathrm{Id} \in L^\infty(\mathbb{R})$ we can write $U(t,\xi) = \bar{U}(t,\xi) + c\chi \circ y(t,\xi)$. Therefore, we can also write (3.6) as

$$y_t = U, \tag{3.19a}$$

$$\bar{U}_t = -Q - c(\chi' \circ y)U, \tag{3.19b}$$

$$c_t = 0, \tag{3.19c}$$

$$h_t = 2(U^2 - P)U_\xi. \tag{3.19d}$$

Theorem 3.3. *Given $X_0 = (\zeta_0, U_0, h_0)$ in E, then there exists a time T depending only on $\|X_0\|_E$ such that* (3.6) *admits a unique solution in $C^1([0,T], E)$ with initial data X_0.*

Proof. Solutions of (3.6) can be rewritten as

$$X(t) = X_0 + \int_0^t F(X(\tau))d\tau,$$

where $F\colon E \to E$ is defined by the right-hand side of (3.6). The integrals are defined as Riemann integrals of continuous functions on the Banach space E. Using Lemma 3.1 we can check that $F(X)$ is a Lipschitz function on bounded sets of E. Since E is a Banach space, we use the standard contraction argument to prove the theorem. ☐

After differentiating (3.6) we obtain

$$y_{\xi,t} = U_\xi, \tag{3.20a}$$

$$\bar{U}_{\xi,t} = \frac{1}{2}h + (U^2 - P)y_\xi - c\chi'' \circ y y_\xi U + c\chi' \circ y Q, \tag{3.20b}$$

$$U_{\xi,t} = \frac{1}{2}h + (U^2 - P)y_\xi, \tag{3.20c}$$

$$h_t = 2(U^2 - P)U_\xi. \tag{3.20d}$$

We define the set $\mathcal{G}$ as follows.

Definition 3.4. *The set $\mathcal{G}$ is composed of all $(\zeta, U, h) \in E$ such that*

$$(\zeta, U) \in [W^{1,\infty}(\mathbb{R})]^2, \quad h \in L^\infty(\mathbb{R}), \tag{3.21a}$$

$$y_\xi \geq 0, \quad h \geq 0, \quad y_\xi + h > 0 \text{ almost everywhere}, \tag{3.21b}$$

$$y_\xi h = U_\xi^2 \text{ almost everywhere}, \tag{3.21c}$$

where we denote $y(\xi) = \xi + \zeta(\xi)$.

As in [12, Lemma 2.7], we can prove that the set $\mathcal{G}$ is preserved by the flow, that is, for any initial data $X_0 = (\zeta_0, U_0, h_0)$ in $\mathcal{G}$, if $X(t) = (\zeta(t), U(t), h(t))$ is the short-time solution of (3.6) in $C^1([0,T], E)$ for some $T > 0$ with initial data (ζ_0, U_0, h_0), then $X(t)$ belongs to $\mathcal{G}$ for all $t \in [0,T]$. Moreover we have that, for almost every $t \in [0,T]$,

$$y_\xi(t,\xi) > 0 \text{ for almost every } \xi \in \mathbb{R}. \tag{3.22}$$

Using this property, we can derive the necessary estimate to prove the global existence of solutions to (3.6).

Theorem 3.5. *For any $X_0 = (y_0, U_0, h_0) \in \mathcal{G}$, the system* (3.6) *admits a unique global solution $X(t) = (y(t), U(t), h(t))$ in $C^1([0,\infty), E)$ with initial data $X_0 = (y_0, U_0, h_0)$. We have $X(t) \in \mathcal{G}$ for all times. If we equip $\mathcal{G}$ with the topology inducted by the E-norm, then the mapping $S : \mathcal{G} \times [0,\infty) \to \mathcal{G}$ defined as*

$$S_t(X_0) = X(t)$$

is a continuous semigroup.

Proof. The solution has a finite time of existence T only if $\|(\zeta, U, h)(t,\,\cdot\,)\|_E$ blows up when t tends to T because, otherwise, by Theorem 3.3, the solution can be prolongated by a small time interval beyond T. Let (ζ, U, h) be a solution of (3.6) in $C^1([0,T), E)$ with initial data (ζ_0, U_0, h_0). We want to prove that

$$\sup_{t\in[0,T)} \|(\zeta(t,\,\cdot\,), U(t,\,\cdot\,), h(t,\,\cdot\,))\|_E < \infty. \tag{3.23}$$

We can follow the proof of [12, Theorem 2.8] once we have established that

$$\sup_{t\in[0,T)} (\|U(t,\cdot)\|_{L^\infty} + \|P(t,\cdot)\|_{L^\infty} + \|Q(t,\cdot)\|_{L^\infty}) < \infty. \tag{3.24}$$

Let us introduce

$$\Gamma = \int_{\mathbb{R}} \bar{U}^2 y_\xi \, d\xi + \|h\|_{L^1}\,.$$

By (3.21c), we have

$$h = U_\xi^2 - \zeta_\xi h \tag{3.25}$$

and therefore $h \in L^1(\mathbb{R})$. Moreover since $h \geq 0$, we have $\|h\|_{L^1} = \int_{\mathbb{R}} h \, d\xi$. We can estimate the $\|\bar{U}\|_{L^\infty}^2$ as follows.

$$\begin{aligned}
\bar{U}^2(\xi) &= 2\int_{-\infty}^{\xi} \bar{U}\bar{U}_\xi \, d\eta \\
&= 2\int_{-\infty}^{\xi} \bar{U}U_\xi \, d\eta - 2\int_{-\infty}^{\xi} c\bar{U}\chi' \circ y y_\xi \, d\eta \\
&\leq \int_{\{\xi | y_\xi(\xi) > 0\}} \bar{U}^2 y_\xi + \frac{U_\xi^2}{y_\xi} \, d\eta - 2\int_{\mathbb{R}} c\bar{U}\chi' \circ y y_\xi \, d\eta \\
&\leq \int_{\{\xi | y_\xi(\xi) > 0\}} (\bar{U}^2 y_\xi + h) \, d\eta + 2C \|\bar{U}\|_{L^\infty} \\
&\leq \Gamma + 2C \|\bar{U}\|_{L^\infty}\,.
\end{aligned}$$

After using that $\|\bar{U}\|_{L^\infty} \leq \frac{1}{4C} \|\bar{U}\|_{L^\infty}^2 + C$, we get

$$\|\bar{U}\|_{L^\infty}^2 \leq 2\Gamma + C. \tag{3.27}$$

From (3.7), we get

$$\begin{aligned}
\|P\|_{L^\infty} &\leq \frac{1}{2}(\|\bar{U}\|_{L^\infty}^2 + 2|c| \|\bar{U}\|_{L^\infty}) \int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|} y_\xi \, d\eta + \Gamma + c^2 \\
&\leq (2\|\bar{U}\|_{L^\infty}^2 + |c|^2) + \Gamma + |c|^2 \\
&\leq 5\Gamma + C.
\end{aligned} \tag{3.28}$$

Similarly, one obtains that

$$\|Q\|_{L^\infty} \leq 5\Gamma + C. \tag{3.29}$$

Hence, (3.24) will be proved when we prove that $\sup_{t\in[0,T)} \Gamma < \infty$. We can now compute the variation of Γ. From (3.19) we get

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \int_{\mathbb{R}} 2\bar{U}\bar{U}_t y_\xi \, d\xi + \int_{\mathbb{R}} \bar{U}^2 y_{\xi t} \, d\xi + \int_{\mathbb{R}} h_t \, d\xi \\
&= \int_{\mathbb{R}} 2\bar{U}(-Q - cU\chi' \circ y) y_\xi \, d\xi + \int_{\mathbb{R}} \bar{U}^2 U_\xi \, d\xi + \int_{\mathbb{R}} 2(U^2 - P) U_\xi \, d\xi.
\end{aligned}$$

We estimate each of these three integrals, that we denote A_1, A_2 and A_3, respectively. We have

$$\begin{aligned}
A_1 &= -2\int_{\mathbb{R}} Q\bar{U} y_\xi d\xi - 2\int_{\mathbb{R}} c\bar{U}^2 \chi' \circ y y_\xi + c^2 \bar{U} \chi \circ y \chi' \circ y y_\xi d\xi \\
&\leq -2\int_{\mathbb{R}} P_\xi \bar{U} d\xi + C\Gamma + C \left\|\bar{U}\right\|_{L^\infty} \\
&\leq 2\int_{\mathbb{R}} P\bar{U}_\xi d\xi + C\Gamma + C \left\|\bar{U}\right\|_{L^\infty},
\end{aligned}$$

after integration by parts, since $P_\xi = Qy_\xi$. We have

$$\begin{aligned}
A_2 &= \int_{\mathbb{R}} \bar{U}^2 \bar{U}_\xi \, d\xi + \int_{\mathbb{R}} c\bar{U}^2 (\chi' \circ y) y_\xi d\xi \\
&= \int_{\mathbb{R}} c\bar{U}^2 (\chi' \circ y) y_\xi d\xi \leq C\Gamma.
\end{aligned}$$

We have

$$\begin{aligned}
A_3 &= 2\int_{\mathbb{R}} U^2 U_\xi d\xi - 2\int_{\mathbb{R}} P\bar{U}_\xi d\xi - 2c\int_{\mathbb{R}} P\chi' \circ y y_\xi d\xi \\
&= -2\int_{\mathbb{R}} P\bar{U}_\xi d\xi - 2c\int_{\mathbb{R}} P\chi' \circ y y_\xi d\xi + \frac{2}{3}c^3 \\
&\leq -2\int_{\mathbb{R}} P\bar{U}_\xi d\xi + C\left\|P\right\|_{L^\infty} + C.
\end{aligned}$$

Finally, by adding up all these estimates, we get

$$\begin{aligned}
\frac{d\Gamma}{dt} &\leq C\Gamma + C\left\|\bar{U}\right\|_{L^\infty} + C\left\|P\right\|_{L^\infty} \\
&\leq C\Gamma + C + C\left\|\bar{U}\right\|_{L^\infty}^2 + C\left\|P\right\|_{L^\infty} \\
&\leq C\Gamma + C,
\end{aligned}$$

by (3.27) and (3.28). Hence, Gronwall's lemma implies that $\sup_{t\in[0,T)} \Gamma(t) < \infty$ Using now (3.27), (3.28), and (3.29), we immediately obtain that the same is true for $\left\|\bar{U}(t,\cdot)\right\|_{L^\infty}$, $\left\|P(t,\cdot)\right\|_{L^\infty}$, and $\left\|Q(t,\cdot)\right\|_{L^\infty}$, which are bounded by a constant only dependent on $\sup_{t\in[0,T)} \Gamma(t) < \infty$ for $t \in [0,T)$. □

4. From Eulerian to Lagrangian coordinates and vice versa

The appropriate set to construct a semigroup of conservative solutions is the set $\mathcal{D}$ defined below, which allows for concentration of the energy in domains of zero measure.

Definition 4.1. *The set $\mathcal{D}$ is composed of all pairs (u,μ) such that $u\in H_{0,\infty}(\mathbb{R})$ and μ is a positive finite Radon measure whose absolutely continuous part, μ_{ac}, satisfies*

$$\mu_{ac}=u_x^2dx. \tag{4.1}$$

The system (3.6) is invariant with respect to relabeling. Relabeling is modeled by the action of the group of diffeomorphisms G that we now define.

Definition 4.2. *We denote by G the subgroup of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$f-\mathrm{Id}\ \text{ and } f^{-1}-\mathrm{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}), \tag{4.2a}$$

$$f_\xi-1 \text{ belongs to } L^2(\mathbb{R}), \tag{4.2b}$$

where Id *denotes the identity function. Given $\alpha>0$, we denote by G_α the subset G_α of G defined by*

$$G_\alpha=\{f\in G\mid \|f-\mathrm{Id}\|_{W^{1,\infty}}+\left\|f^{-1}-\mathrm{Id}\right\|_{W^{1,\infty}}\leq\alpha\}. \tag{4.3}$$

We define the subsets $\mathcal{F}_\alpha$ and $\mathcal{F}$ of $\mathcal{G}$ as follows

$$\mathcal{F}_\alpha=\{X=(y,U,h)\in\mathcal{G}\mid y+H\in G_\alpha\},$$

and

$$\mathcal{F}=\{X=(y,U,h)\in\mathcal{G}\mid y+H\in G\},$$

where $H(t,\xi)$ is defined by

$$H(t,\xi)=\int_{-\infty}^{\xi}h(t,\tau)d\tau,$$

which is finite since, from (3.21c), we have $h=U_\xi^2-\zeta_\xi h$ and therefore $h\in L^1(\mathbb{R})$. For $\alpha=0$, we have $G_0=\{\mathrm{Id}\}$. As we shall see, the space $\mathcal{F}_0$ will play a special role. These sets are relevant only because they are preserved by the governing equation (3.6). In particular, while the mapping $\xi\mapsto y(t,\xi)$ may not be a diffeomorphism for some time t, the mapping $\xi\mapsto y(t,\xi)+H(t,\xi)$ remains a diffeomorphism for all times t. As in [12, Lemma 3.3], we can establish that the space $\mathcal{F}$ is preserved by the governing equations (3.6). More precisely, we have that, given α, $T\geq 0$, and $X_0\in\mathcal{F}_\alpha$,

$$S_t(X_0)\in\mathcal{F}_{\alpha'},$$

for all $t\in[0,T]$ where α' only depends on T, α and $\|X_0\|_E$. For the sake of simplicity, for any $X=(y,U,h)\in\mathcal{F}$ and any function $f\in G$, we denote $(y\circ f, U\circ f, h\circ f f_\xi)$ by $X\circ f$. Then, $X\circ f$ corresponds to the relabeling of X with respect to the relabeling function $f\in G$. The map from $G\times\mathcal{F}$ to $\mathcal{F}$ given by $(f,X)\mapsto X\circ f$ defines an action of the group G on $\mathcal{F}$, see [12, Proposition 3.4]. Since G is acting on $\mathcal{F}$, we can consider the quotient space $\mathcal{F}/G$ of $\mathcal{F}$ with respect to the action of the group G. The equivalence relation on $\mathcal{F}$ is defined as follows: For any $X,X'\in\mathcal{F}$, we say that X and X' are equivalent if there exists $f\in G$ such that $X'=X\circ f$. We denote by $\Pi(X)=[X]$ the projection of $\mathcal{F}$ into the quotient space $\mathcal{F}/G$, and introduce the mapping $\Gamma\colon\mathcal{F}\to\mathcal{F}_0$ given by

$$\Gamma(X)=X\circ(y+H)^{-1}$$

for any $X=(y,U,h)\in\mathcal{F}$. We have $\Gamma(X)=X$ when $X\in\mathcal{F}_0$ and Γ is invariant under the G action, that is, $\Gamma(X\circ f)=\Gamma(X)$ for any $X\in\mathcal{F}$ and $f\in G$. Hence,

we can define a mapping $\tilde{\Gamma}$ from the quotient space $\mathcal{F}/G$ to $\mathcal{F}_0$ as $\tilde{\Gamma}([X]) = \Gamma(X)$ and the sets $\mathcal{F}_0$ and $\mathcal{F}$ are in bijection as

$$\tilde{\Gamma} \circ \Pi|_{\mathcal{F}_0} = \mathrm{Id}\,|_{\mathcal{F}_0}.$$

We equip $\mathcal{F}_0$ with the metric induced by the E-norm, i.e., $d_{\mathcal{F}_0}(X, X') = \|X - X'\|_E$ for all $X, X' \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is closed in E, this metric is complete. We define the metric on $\mathcal{F}/G$ as

$$d_{\mathcal{F}/G}([X], [X']) = \|\Gamma(X) - \Gamma(X')\|_E\,,$$

for any $[X], [X'] \in \mathcal{F}/G$. Then, $\mathcal{F}/G$ is isometrically isomorphic with $\mathcal{F}_0$ and the metric $d_{\mathcal{F}/G}$ is complete.

We denote by $S\colon \mathcal{F} \times [0, \infty) \to \mathcal{F}$ the continuous semigroup which to any initial data $X_0 \in \mathcal{F}$ associates the solution $X(t)$ of the system of differential equations (3.6) at time t. As indicated earlier, the Camassa–Holm equation is invariant with respect to relabeling. More precisely, using our terminology, we obtain the diagram

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, "\Pi"] & \mathcal{F}/G \\
\mathcal{F}_\alpha \arrow[u, "\Gamma"] & \\
\mathcal{F}_0 \arrow[u, "S_t"] \arrow[r, "\Pi"] & \mathcal{F}/G \arrow[uu, "\tilde{S}_t"']
\end{tikzcd} \tag{4.4}$$

which summarizes the following theorem.

Theorem 4.3. *For any $t > 0$, the mapping $S_t\colon \mathcal{F} \to \mathcal{F}$ is G-equivariant, that is,*

$$S_t(X \circ f) = S_t(X) \circ f \tag{4.5}$$

for any $X \in \mathcal{F}$ and $f \in G$. Hence, the mapping $\tilde{S}_t$ from $\mathcal{F}/G$ to $\mathcal{F}/G$ given by

$$\tilde{S}_t([X]) = [S_t X]$$

is well-defined. It generates a continuous semigroup.

Proof. See [12, Theorem 3.7]. ☐

Note that the continuity of $\tilde{S}_t$ holds because, for any given $\alpha \geq 0$, the restriction of Γ to $\mathcal{F}_\alpha$ is a continuous mapping from $\mathcal{F}_\alpha$ to $\mathcal{F}_0$, see [12, Lemma 3.5].

Our next task is to derive the correspondence between Eulerian coordinates (functions in $\mathcal{D}$) and Lagrangian coordinates (functions in $\mathcal{F}/G$). Let us denote by $L\colon \mathcal{D} \to \mathcal{F}/G$ the mapping transforming Eulerian coordinates into Lagrangian coordinates whose definition is contained in the following theorem.

Definition 4.4. *For any (u, μ) in $\mathcal{D}$, let*

$$y(\xi) = \sup\{y \mid \mu((-\infty, y)) + y < \xi\}\,, \tag{4.6a}$$

$$U(\xi) = u \circ y(\xi), \tag{4.6b}$$

$$h(\xi) = 1 - y_\xi(\xi). \tag{4.6c}$$

Then $(y, U, h) \in \mathcal{F}_0$. We define $L(u, \mu) = (y, U, h)$.

Note that the mapping L depends on the partition function χ. The well-posedness of this definition is established in the same way as in [12, Theorem 3.8]. In the other direction, we obtain μ, the energy density in Eulerian coordinates, by pushing forward by y the energy density in Lagrangian coordinates $hd\xi$. We are led to the mapping M which transforms Lagrangian coordinates into Eulerian coordinates and whose definition is contained in the following theorem.

Definition 4.5. *Given any element X in $\mathcal{F}/\mathcal{G}$. Then (u,μ) defined as follows*

$$u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \tag{4.7a}$$

$$\mu = y_{\#}(h\,d\xi) \tag{4.7b}$$

belongs to $\mathcal{D}$. We denote by $M\colon \mathcal{F}/\mathcal{G} \to \mathcal{D}$ the mapping which to any $X \in \mathcal{F}/\mathcal{G}$ associates (u,μ) as given by (4.7).

The well-posedness of this definition is established in the same way as in [12, Theorem 3.11]. Finally, one can show that the transformation from Eulerian to Lagrangian coordinates is a bijection (see [12, Theorem 3.12]).

Theorem 4.6. *The mappings M and L are invertible. We have*

$$L \circ M = \mathrm{Id}_{\mathcal{F}/G} \text{ and } M \circ L = \mathrm{Id}_{\mathcal{D}}.$$

5. Continuous semigroup of solutions on $\mathcal{D}$

For each $t \in \mathbb{R}$, we define the mapping T_t from $\mathcal{D}$ to $\mathcal{D}$ as

$$T_t = M\tilde{S}_t L. \tag{5.1}$$

It corresponds to the following diagram:

$$\begin{tikzcd} \mathcal{D} & \mathcal{F}/G \arrow[l, "M"'] \\ \mathcal{D} \arrow[u, "T_t"] \arrow[r, "L"] & \mathcal{F}/G \arrow[u, "\tilde{S}_t"'] \end{tikzcd}$$

We define global weak conservative solution to the Camassa-Holm equation as follows.

Definition 5.1. *Assume that $u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ satisfies*
(i) $u \in L^\infty_{\mathrm{loc}}([0,\infty), H_\infty(\mathbb{R}))$,
(ii) the equations

$$\iint_{[0,\infty)\times\mathbb{R}} \Big[-u(t,x)\phi_t(t,x) + \big(u(t,x)u_x(t,x) + P_x(t,x)\big)\phi(t,x)\Big]dxdt = \int_{\mathbb{R}} u(0,x)\phi(x,0)dx \tag{5.2}$$

and

$$\iint_{[0,\infty)\times\mathbb{R}} \Big[(P(t,x) - u^2(t,x) - \frac{1}{2}u_x^2(t,x))\phi(t,x) + P_x(t,x)\phi_x(t,x)\Big]dxdt = 0, \tag{5.3}$$

hold for all $\phi \in C_0^\infty([0,\infty)\times\mathbb{R})$. Then we say that u is a weak global solution of the Camassa–Holm equation. If u in addition satisfies

$$(u^2 + u_x^2)_t + (u(u^2+u_x^2))_x - (u^3 - 2Pu)_x = 0$$

in the sense that

(5.4)
$$\iint_{(0,\infty)\times\mathbb{R}} \Big[(u^2(t,x)+u_x^2(t,x))\phi_t(t,x)+(u(t,x)(u^2(t,x)+u_x^2(t,x)))\phi_x(t,x) \\ -(u^3(t,x)-2P(t,x)u(t,x))\phi_x(t,x)\Big]dxdt=0,$$

for any $\phi\in C_0^\infty((0,\infty)\times\mathbb{R})$, *we say that* u *is a weak global conservative solution of the Camassa–Holm equation.*

On $\mathcal{D}$ we define the distance $d_{\mathcal{D}}$ which makes the bijection L between $\mathcal{D}$ and $\mathcal{F}/G$ into an isometry:

$$d_{\mathcal{D}}((u,\mu),(\bar u,\bar\mu))=d_{\mathcal{F}/G}(L(u,\mu),L(\bar u,\bar\mu)).$$

Since $\mathcal{F}/G$ equipped with $d_{\mathcal{F}/G}$ is a complete metric space, the space $\mathcal{D}$ equipped with the metric d_D is a complete metric space. Our main theorem then reads as follows.

Theorem 5.2. *The semigroup* $(T_t,d_{\mathcal{D}})$ *is a continuous semigroup on* $\mathcal{D}$ *with respect to the metric* $d_{\mathcal{D}}$. *Moreover, given any intial condition* $(u_0,\mu_0)\in\mathcal{D}$, *we denote* $(u,\mu)(t)=T_t(u_0,\mu_0)$. *Then* $u(t,x)$ *is a weak global conservative solution of the Camassa–Holm equation. Moreover, letting* $\nu=u^2\,dx+\mu$, *we have*

$$\nu_t+(u\nu)_x-(u^3-2Pu)_x=0$$

in the sense of distribution, that is,

$$\iint_{[0,\infty)\times\mathbb{R}}(\phi_t(t,x)+u(t,x)\phi_x(t,x))d\nu(t,x)dt \\ -\iint_{[0,\infty)\times\mathbb{R}}(u^3(t,x)-2P(t,x)u(t,x))\phi_x(t,x)dxdt=-\int_{\mathbb{R}}\phi(0,x)d\nu(0,x)dx. \tag{5.5}$$

Proof. First we prove that T_t is a semigroup. Since $\tilde S_t$ is a mapping from $\mathcal{F}_0$ to $\mathcal{F}_0$, we have

$$T_tT_{t'}=M\tilde S_tLM\tilde S_{t'}L=M\tilde S_t\tilde S_{t'}L=M\tilde S_{t+t'}L=T_{t+t'},$$

where we used (5.1) and the semigroup property of $\tilde S_t$. To show that $u(t,x)$ is a weak global solutions, we have to show that (5.2) and (5.3) are satisfied. The proof of (5.2) and (5.3) is essentially the same as in [12]. Let us check that (5.5) is fullfilled. After making the change of variable $x=y(t,\xi)$ we obtain

$$\begin{aligned}\iint_{[0,\infty)\times\mathbb{R}}u^2(t,x)\phi_t(t,x)dxdt&=\iint_{[0,\infty)\times\mathbb{R}}U^2(t,\xi)\phi_t(t,y(t,\xi))y_\xi(t,\xi)d\xi dt\\ &=\iint_{[0,\infty)\times\mathbb{R}}U^2(t,\xi)\Big((\phi(t,y(t,\xi)))_t-\phi_x(t,y(t,\xi))y_t(t,\xi)\Big)y_\xi(t,\xi)d\xi dt\\ &=-\int_{\mathbb{R}}U^2(0,\xi)\phi(0,y(0,\xi))d\xi-\iint_{[0,\infty)\times\mathbb{R}}U^3(t,\xi)\phi_\xi(t,y(t,\xi))d\xi dt\\ &\quad+\iint_{[0,\infty)\times\mathbb{R}}\Big(2U(t,\xi)Q(t,\xi)y_\xi(t,\xi)-U^2(t,\xi)U_\xi(t,\xi)\Big)\phi(t,y(t,\xi))d\xi dt\\ &=-\int_{\mathbb{R}}u^2(0,x)\phi(0,x)dx\end{aligned}$$

$$+\iint_{[0,\infty)\times\mathbb{R}}\Big(2u(t,x)P_x(t,x)-u^2(t,x)u_x(t,x)\Big)\phi(t,x)-u^3(t,x)\phi_x(t,x)dxdt,$$

and

$$\begin{aligned}
\iint_{[0,\infty)\times\mathbb{R}}\phi_t(t,x)d\mu(t,x)dt &= \iint_{[0,\infty)\times\mathbb{R}}\phi_t(t,y(t,\xi))h(t,\xi)d\xi dt\\
&= \iint_{[0,\infty)\times\mathbb{R}}\Big((\phi(t,y(t,\xi)))_t-\phi_x(t,y(t,\xi))y_t(t,\xi)\Big)h(t,\xi)d\xi dt\\
&= -\int_{\mathbb{R}}\phi(0,y(0,\xi))h(0,\xi)d\xi\\
&\quad -\iint_{[0,\infty)\times\mathbb{R}}h_t(t,\xi)\phi(t,y(t,\xi))+U(t,\xi)h(t,\xi)\phi_x(t,y(t,\xi))d\xi dt\\
&= -\int_{\mathbb{R}}\phi(0,y(0,\xi))h(0,\xi)d\xi\\
&\quad -\iint_{[0,\infty)\times\mathbb{R}}2(U^2(t,\xi)-P(t,\xi))U_\xi(t,\xi)\phi(t,y(t,\xi))+U(t,\xi)h(t,\xi)\phi_x(t,y(t,\xi))d\xi dt\\
&= -\int_{\mathbb{R}}\phi(0,x)d\mu(0,x)dx-\iint_{[0,\infty)\times\mathbb{R}}\phi_x(t,x)u(t,x)d\mu(t,x)dt\\
&\quad -\iint_{[0,\infty)\times\mathbb{R}}2(u^2(t,x)-P(t,x))u_x(t,x)\phi(t,x)dxdt.
\end{aligned}$$

This finishes the proof. Since for almost every $t\in[0,T]$, $y_\xi(t,\xi)>0$ for almost every $\xi\in\mathbb{R}$, see (3.22), the property (5.4) follows from (5.5). □

6. Invariance of the topology with respect to the choice of the partition function

The mappings L and M depend on the choice of the partition function χ. To emphasize this dependence, we write L_χ and M_χ. In this section we prove that different choices of the partition function χ lead to the same topology in $\mathcal{D}$. Given two partition functions χ and $\tilde\chi$, we obtain two topologies

$$d_{\mathcal{D}}((u,\mu),(\bar u,\bar\mu))=\|L(u,\mu)-L(\bar u,\bar\mu)\|_{E_\chi}, \tag{6.1}$$

and

$$\tilde d_{\mathcal{D}}((u,\mu),(\bar u,\bar\mu))=\|L(u,\mu)-L(\bar u,\bar\mu)\|_{E_{\tilde\chi}}. \tag{6.2}$$

In (6.1) and (6.2), we add the subscripts χ and $\tilde\chi$ to indicate which norm is used on E.

Theorem 6.1. *We consider two partition functions χ and $\tilde\chi$. Then, the metric they induce on $\mathcal{D}$ are equivalent, that is, there exists a constant $C>0$ which only depends on χ and $\tilde\chi$ such that*

$$\frac{1}{C}\tilde d_D((\bar u,\bar\mu),(u,\mu))\le d_D((\bar u,\bar\mu),(u,\mu))\le C\tilde d_D((\bar u,\bar\mu),(u,\mu))$$

for any $(\bar u,\bar\mu)$ and (u,μ) in $\mathcal{D}$.

Proof. Let $(y,U,h)=L(u,\mu)$ and $(\bar y,\bar U,\bar h)=L(\bar u,\bar\mu)$. We have

$$\|L(u,\mu)-L(\bar u,\bar\mu)\|_{E_{\tilde\chi}}=\left\|(\zeta,I_{\tilde\chi}^{-1}(U),h)-(\bar\zeta,I_{\tilde\chi}^{-1}(\bar U),\bar h)\right\|_{\bar E}$$

$$= \left\|\zeta - \bar{\zeta}\right\|_V + \left\|I_{\bar{\chi}}^{-1}(U - \bar{U})\right\|_{H^1(\mathbb{R})\times\mathbb{R}} + \left\|h - \bar{h}\right\|_{L^2}$$
$$\leq \left\|\zeta - \bar{\zeta}\right\|_V + C\left\|\Psi^{-1}I_{\bar{\chi}}^{-1}(U - \bar{U})\right\|_{H^1(\mathbb{R})\times\mathbb{R}} + \left\|h - \bar{h}\right\|_{L^2},$$

see (2.5) for the definition of Ψ. The linear mapping Ψ is continuous and C is its operator norm, which only depends on χ and $\bar{\chi}$. Hence, since $I_\chi = I_{\bar{\chi}} \circ \Psi$,

$$\begin{aligned}
\|L(u,\mu) - L(\bar{u},\bar{\mu})\|_{E_{\bar{\chi}}} &\leq \left\|\zeta - \bar{\zeta}\right\|_V + C\left\|I_{\chi}^{-1}(U - \bar{U})\right\|_{H^1(\mathbb{R})\times\mathbb{R}} + \left\|h - \bar{h}\right\|_{L^2} \\
&\leq C\left\|(\zeta, I_\chi^{-1}(U), h) - (\bar{\zeta}, I_\chi^{-1}(\bar{U}), \bar{h})\right\|_{\bar{E}} \\
&= C\left\|L(u,\mu) - L(\bar{u},\bar{\mu})\right\|_{E_\chi}.
\end{aligned}$$

□

The metric $d_{\mathcal{D}}$ on $\mathcal{D}$ gives the structure of a complete metric space while it makes the semigroup T_t of conservative solutions continuous for the Camassa–Holm equation. In that respect, it is a suitable metric for the Camassa–Holm equation. The definition of $d_{\mathcal{D}}$ is not straightforward but it can be compared with more standard topologies. We have that the mapping

$$u \mapsto (u, u_x^2\,dx)$$

is continuous from $H_{0,\infty}(\mathbb{R})$ into $\mathcal{D}$. In other words, given a sequence $u_n \in H_{0,\infty}(\mathbb{R})$ which converges to u in $H_{0,\infty}(\mathbb{R})$, then $(u_n, u_{n,x}^2\,dx)$ converges to $(u, u_x^2\,dx)$ in $\mathcal{D}$. See [12, Proposition 5.1]. Conversely, let (u_n,μ_n) be a sequence in $\mathcal{D}$ that converges to (u,μ) in $\mathcal{D}$. Then

$$u_n \to u \text{ in } L^\infty(\mathbb{R}) \quad \text{and} \quad \mu_n \overset{*}{\rightharpoonup} \mu.$$

See [12, Proposition 5.2].

7. Conservative solutions with vanishing asymptotics

In this section we want to clarify the connection between the approach used here in the case $c_- = c_+ = 0$ and the one used in [12], which also answers the questions why the proofs are quite similar and why we speak of conservative solutions.

Theorem 7.1. *Let (u_0,μ_0) be a pair of Eulerian coordinates as in Definition* 4.1*, and $(\tilde{u}_0,\tilde{\mu}_0)$ the pair of Eulerian coordinates as defined in* [12, Definition 3.1]*, such that $u_0(x) = \tilde{u}_0(x)$ and such that*

$$\tilde{\mu}_0((-\infty,x)) - \mu_0((-\infty,x)) = \int_{-\infty}^x u_0(x)^2 dx, \quad x \in \mathbb{R}.$$

Then the solutions $(u(t),\mu_t)$ and $(\tilde{u}(t),\tilde{\mu}_t)$ satisfy $u(t,x) = \tilde{u}(t,x)$ and

$$\tilde{\mu}_t((-\infty,x)) - \mu_t((-\infty,x)) = \int_{-\infty}^x u(t,x)^2 dx, \quad x \in \mathbb{R}.$$

Proof. Let (u_0,μ_0) be a pair of Eulerian coordinates as defined in Definition 4.1, and $(u_0,\tilde{\mu}_0)$ the set of Eulerian coordinates defined as in [12, Definition 3.1], such that

$$\tilde{\mu}_0((-\infty,x)) - \mu_0((-\infty,x)) = \int_{-\infty}^x u_0(x)^2 dx, \quad x \in \mathbb{R}.$$

Moreover, let (y,U,h) be the Lagrangian coordinates to (u,μ) and define

$$\hat{H}(t,\xi) = \int_{-\infty}^{\xi} h + U^2 y_\xi d\xi.$$

Then by construction (cf. Definition 4.4), we have

$$y(0,\xi)+\hat{H}(0,\xi)=\int_{-\infty}^{\xi} U^2 y_\xi(0,\xi)d\xi+\xi.$$

The right-hand side belongs to $\mathcal{G}$, the set of relabeling functions, which coincides with the one used in [12]. Thus let $f(\xi)=\int_{-\infty}^{\xi} U^2 y_\xi(0,\xi)d\xi+\xi$. Then we have for almost every $\xi\in\mathbb{R}$, that

$$y_0(\xi)+\mu_0((-\infty,y_0(\xi)))=\xi,$$

which implies

$$y_0(\xi)+\tilde{\mu}_0((-\infty,y_0(\xi)))=f(\xi).$$

Thus setting $\tilde{y}_0(\xi)=y_0(f^{-1}(\xi))$ yields

$$\tilde{y}_0(\xi)+\tilde{\mu}_0((-\infty,\tilde{y}_0(\xi)))=\xi.$$

Moreover, one can conclude that

$$\tilde{y}_0(\xi)=y_0(f^{-1}(\xi)),\qquad \text{for all } x\in\mathbb{R},$$

by using Definition 4.4 and [12, Theorem 3.8]. Analogously one can proceed for the other involved functions so that $X_0\circ f^{-1}=\tilde{X}_0$.

Furthermore, from (3.6) we can conclude that the variables (y,U,H) satisfy

$$y_t=U, \tag{7.1a}$$

$$U_t=-Q, \tag{7.1b}$$

$$\hat{H}_t=U^3-2PU, \tag{7.1c}$$

which coincides with the system of ordinary differential equations for the Lagrangian coordinates considered in [12]. In addition we know from [12, Theorem 3.7] that $S_t(X\circ f)=S_t(X)\circ f$, and therefore using that the mapping from Lagrangian to Eulerian coordinates (cf. [12, Theorem 3.11]) is independent of the element of the equivalence class we choose, we obtain that the pairs (u_0,μ_0) and $(u_0,\tilde{\mu}_0)$ with $\tilde{\mu}_0((-\infty,x))-\mu_0((-\infty,x))=\int_{-\infty^x} u_0^2(x)dx$, give rise to the same conservative solution. □

References

[1] M. Bendahmane, G. M. Coclite and K. H. Karlsen H^1-perturbations of smooth solutions for a weakly dissipative hyperelastic-rod wave equation *Mediterranean Journal of Mathematics* 3:419–432, 2006.

[2] A. Bressan and A. Constantin. Global conservative solutions of the Camassa–Holm equation. *Arch. Ration. Mech. Anal.* 183:215–239, 2007.

[3] A. Bressan and A. Constantin. Global dissipative solutions of the Camassa–Holm equation. *Analysis and Applications* 5:1–27, 2007.

[4] A. Bressan, H. Holden, and X. Raynaud. Lipschitz metric for the Hunter–Saxton equation. *J. Math. Pures Appl.* 94:68–92, 2010.

[5] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solutions. *Phys. Rev. Lett.* 71(11):1661–1664, 1993.

[6] R. Camassa, D. D. Holm, and J. Hyman. A new integrable shallow water equation. *Adv. Appl. Mech.* 31:1–33, 1994.

[7] A. Constantin. Finite propagation speed for the Camassa–Holm equation. *J. Math. Phys.* 41:023506, 2005.

[8] A. Constantin and J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.* 181:229–243, 1998.

[9] A. Constantin and D. Lannes. The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. *Arch. Rat. Mech. Anal.* 192:165–186, 2009.

[10] K. Grunert, H. Holden, and X. Raynaud. Lipschitz metric for the periodic Camassa–Holm equation. *J. Differential Equations* 250:1460–1492, 2011.

[11] K. Grunert, H. Holden, and X. Raynaud. Lipschitz metric for the Camassa–Holm equation on the line. *Preprint*, submitted.

[12] H. Holden and X. Raynaud. Global conservative solutions for the Camassa–Holm equation — a Lagrangian point of view. *Comm. Partial Differential Equations* 32:1511–1549, 2007.

[13] H. Holden and X. Raynaud. Global conservative solutions of the generalized hyperelastic-rod wave equation. *J. Differential Equations* 233(2):448–484, 2007.

[14] H. Holden and X. Raynaud. Dissipative solutions of the Camassa–Holm equation. *Discrete Contin. Dyn. Syst.* 24:1047–1112, 2009.

[15] H. Holden and X. Raynaud. Periodic conservative solutions of the Camassa–Holm equation. *Ann. Inst. Fourier (Grenoble)* 58:945–988, 2008.

[16] J. Lenells. Travelling wave solutions of the Camassa–Holm equation. *J. Differential Equations* 217:393–430, 2005.

[17] J. Lenells. Classification of all travelling-wave solutions for some nonlinear dispersive equations. *Phil. Trans. R. Soc. A* 365:2291–2298, 2007.

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: katrin.grunert@univie.ac.at
URL: http://www.mat.univie.ac.at/~grunert/

Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway, and Centre of Mathematics for Applications, University of Oslo, NO-0316 Oslo, Norway
E-mail address: holden@math.ntnu.no
URL: http://www.math.ntnu.no/~holden/

Centre of Mathematics for Applications, University of Oslo, NO-0316 Oslo, Norway
E-mail address: xavierra@cma.uio.no
URL: http://folk.uio.no/xavierra/